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Paired-domination of Trees

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Abstract. Let $G=(V,E)$ be a graph without isolated vertices. A set $S\subseteq V$ is a *paired-dominating set* if it dominates V and the subgraph induced by S, $\langle S\rangle$, contains a perfect matching. The paired-domination number $\gamma_p(G)$ is defined to be the minimum cardinality of a paired-dominating set S in G. In this paper, we present a linear-time algorithm computing the paired-domination number for trees and characterize trees with equal domination and paired-domination numbers.

1. Introduction

Let $G=(V,E)$ be a graph without isolated vertices with vertex set V of order n and edge set E. Consider a vertex $v\in V$. The *open neighborhood* of v is defined by $N(v)=\{u\in V \mid uv\in E\}$ and the *closed neighborhood of* v is defined by $N[v]=N(v)\cup\{v\}$. For $S\subseteq V$, the open neighborhood of S is the union of the open neighborhoods of vertices in S, that is, $N(S)=\cup_{v\in S}N(v)$ and the closed neighborhood of S is defined similarly by $N[S]=\cup_{v\in S}N[v]$. The *private neighborhood* $PN(v,S)$ of $v\in S$ is defined by $PN(v,S)=N[v]-N[S-\{v\}]$. The subgraph of G induced by the vertices in S is denoted by $\langle S\rangle$.

In a tree T, a vertex is *remote* if it is adjacent to a leaf and is a *branch vertex* if it has the degree at least 3. Denote by $B(T)$ the set of branch vertices of T and by $L(T)$ the set of leaves.

A set $S\subseteq V$ is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A minimum dominating set of a graph G is also called a $\gamma(G)$-set of G. If X dominates $Y\subseteq V$, we write $X\succ Y$, or $X\succ G$ if $Y=V$, or $X\succ y$ if $Y=\{y\}$. The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set. A set $S\subseteq V$ is a *restrained dominating set*, if every vertex not in S is adjacent to a vertex in S and to a vertex in $V-S$. The *restrained domination number* of G, noted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G.

A set $S\subseteq V$ is a *paired-dominating* set if S dominates V and the induced subgraph $\langle S\rangle$ has a perfect matching. If $v_jv_k=e_i\in M$, where M is a perfect matching of $\langle S\rangle$, we say that v_j and v_k are paired in S. Paired-domination was introduced by Haynes and Slater [1] with the following application in mind.

If we think of each $s \in S \subseteq V$ as the location of a guard capable of protecting each vertex in $N[s]$, then domination requires every vertex to be protected. In paired-domination, each guard is assigned another adjacent one, and they are designed as backup for each other.

The *paired-domination number* $\gamma_p(G)$ is defined to be the minimum cardinality of a paired-dominating set S in G. A minimum paired-dominating set of a graph G is called a $\gamma_p(G)$-set of G.

OBSERVATION 1. [1] *For any graph G without isolated vertices,* $\gamma(G) \leqslant \gamma_p(G) \leqslant 2\beta_1(G)$ *and* $\gamma_p(G)$ *is even, where* $\beta_1(G)$ *denotes the size of a maximum independent set of edges.*

THEOREM 1. [1] *Deciding, for a given graph H and a positive (even) integer* $k \leqslant |V(H)|$, *"Is* $\gamma_p(H) \leqslant k$?*" is NP-complete.*

Since the problem of determining the paired-domination number of an arbitrary graph is NP-hard, it is theoretically important to consider algorithms of paired-domination number in special graphs. In another aspect, an area of research that has received considerable attention is the study of classes of graphs for which some of these parameters are equal or not equal. For any graph theoretical parameters λ and u, we define G to be a (λ, u)-graph if $\lambda(G) = u(G)$. The class of (γ, i)-trees, that is, trees for which $\gamma = i$, was characterized in [5]. Several classes of (γ, i)-graphs have been found (see [6]). A constructive characterization of trees with equal independent domination and restrained domination numbers was given in [4]. In Section 2, we present a linear time algorithm for computing paired-domination number in trees. A characterization of trees with equal domination and paired-domination numbers is given in the third section.

2. A paired-domination algorithm of trees

We now define some basic concepts and notations for trees. A *rooted tree T* is a directed tree in which there exists a vertex r with the property that there is a directed path in T from r to every other vertex in T. The vertex r is unique with respect to the above-mentioned property and is called the *root* of T. For a vertex v of a rooted tree T, the *parent* $p(v)$ of v is the unique vertex such that there is a directed edge from $p(v)$ to v, a *child* of v is a vertex u such that $p(u) = v$, and a *descendant* of v is a vertex u such that there is a directed $v - u$ path in T. Also, for a directed tree, the open neighborhood of a vertex v is defined as $N(v) = \{u \in V \mid uv \in E \text{ or } vu \in E\}$. That is, the parent and the children of v detemine the open neighborhood of v. The other definitions for a graph given in the introduction are the same for a directed tree.

When we consider a rooted tree, we will assume its edges to be directed as explained above, but will not mention this explicitly. We will also refer to the ''edges'' of the tree, not ''arcs'' or ''directed edges''. We define the following notations:

$C(v) = \{u \in V : u \text{ is a child of } v\}$

$c(v) = u \text{ if } C(v) = u$

$D(v) = \{u \in V : u \text{ is a descendant of } v\}$

$D[v] = D(v) \cup \{v\}$

The subtree of T induced by $D[v]$ is denoted by T_v. Note that if T is rooted at v, then $T = T_v$.

A path P in T is said to be a $v - L$ path if P joins v to a leaf of T. P_l represents a path with l vertices. The length of a path P is defined as the number of edges in that path, and is denoted by $l(P)$. Consider T to be rooted and for $j = 0, 1, 2, 3$ define

$$C^j(v) = \{u \in C(v) : T_u \text{ contains a } u - L \text{ path } P \text{ with } l(P) = j(mod\ 4)\}$$

Suppose T is rooted at v, i.e., $T = T_v$. Let u be a branch vertex at maximum distance from v. Note that $|C(u)| \geqslant 2$ and $d(x) \leqslant 2$ for each $x \in D(u)$, where $d(x)$ denotes the degree of vertex x. For each $w \in C(u)$, we allocate a priority to w, where $w^1 \in C^1(u)$ have higher priority than $w^0 \in C^0(u)$, $w^0 \in C^0(u)$ have higher priority than $w^2 \in C^2(u)$, which again have higher priority than $w^3 \in C^3(u)$.

Next we present a linear time algorithm for the minimum paired-domination problem of trees. The algorithm first breaks the original tree T_v into a collection of components, T_0, where each component is a path (Step 3 and Step 4). Then, for each path in T_0 the minimum paired-dominating set is computed (Step 7 to Step 14).

Algorithm 1. Minimum paired-domination for trees.

Input. A rooted tree T_v with root v such that $|V(T_v)| \geqslant 2$.

Step 1: Set $T := T_v$, $J := \emptyset$, $S := \emptyset$. We also set T_0 to be a dummy empty graph, i.e., a graph with no vertices and no edges.

Step 2: Use the breadth-first method to search all the vertices of T_v, determine the distance $d(v, x)$ for each vertex $x \in V(T_v)$ and simultaneously generate the branch-vertex sequence

$$(u_1, u_2, \ldots, u_r)$$

such that each branch-vertex appears exactly once in the sequence and such that

$$d(v, u_1) \leqslant d(v, u_2) \leqslant \cdots \leqslant d(v, u_r)\ .$$

Set $B(T) := \{u_1, u_2, \ldots, u_r\}$ and $m := r$.

Step 3: If $m = 0$ (i.e., $B(T) = \emptyset$), set $T_0 := T_0 \cup T$, go to Step 5. Otherwise (i.e., $B(T) \neq \emptyset$), go to Step 4.

Step 4: Set $u := u_m$. For each child x of u, let x' be the unique leaf in T_x. For $i = 0, 1, 2, 3$ set

$$C^j(u) := \{x \in C(u) : d(x, x') = j \pmod 4\}\,.$$

Choose a child z of u such that z has the highest priority in T_u. Set

$$T := T - \cup_{w \in C(u)-z} D[w]\,,$$

$$\begin{aligned} T_0 := T_0 &\cup (\cup_{w \in (C(u)-z)-C^0(u)} T_w) \\ &\cup (\cup_{w \in (C(u)-z)\cap C^0(u), c(w)\neq\emptyset} (T_w - wc(w))) \\ &\cup (\cup_{w \in (C(u)-z)\cap C^0(u), c(w)=\emptyset} \{w\})\,. \end{aligned}$$

Furthermore, if $(C(u) - z) \cap C^0(u) \neq \emptyset$, we label the vertex u by $*$ and set $J := J \cup \{u\}$. Then set $B(T) := B(T) - u$, $m := m - 1$ and go to Step 3.

Step 5: Set T_{10} to be the isolated vertex set of the graph T_0, set $T_{20} := T_0 - T_{10}$.

Step 6: If T_{20} is a dummy empty graph, then stop. Otherwise, go to Step 7.

Step 7: Arbitrarily choose a component P of T_{20}. Clearly, P is a directed path. We suppose that

$$P = v_{i_1} v_{i_2} \ldots v_{i_2} \quad (v_{il} \not\in J)\,.$$

Set $T' := P$, $k := l$.

Step 8: If $k = 2$, set $S := S \cup V(P)$, go to Step 14. Otherwise ($k \geqslant 3$), go to Step 9.

Step 9: If $v_{i_1} \not\in J$, set $S := S \cup \{v_{i_2}, v_{i_3}\}$ then go to Step 10. Otherwise ($v_{i_1} \in J$), set $S := S \cup \{v_{i_1}, v_{i_2}\}$, go to Step 11.

Step 10: If $k = 3$, go to Step 14. Otherwise ($k \geqslant 4$), if $v_{i_4} \in J$, set $T' := T' - \{v_{i_1}, v_{i_2}, v_{i_3}\}$, $v_{i_j} := v_{i_{j+3}}$ for $1 \leqslant j \leqslant k - 3$, $k := k - 3$, go to Step 8. If $v_{i_4} \not\in J$, go to Step 13.

Step 11: If $v_{i_3} \in J$, set $T' := T' - \{v_{i_1}, v_{i_2}\}$, $v_{i_j} := v_{i_{j+2}}$ for $1 \leqslant \mathrm{j} \leqslant \mathrm{k} - 2$, $\mathrm{k} := \mathrm{k} - 2$, go to Step 8; if $v_{i_3} \not\in J$, go to Step 12.

Step 12: If $k = 3$, go to Step 14. If $k = 4$, set $S := S \cup \{v_{i_3}, v_{i_4}\}$, go to Step 14. If $k \geqslant 5$, set $T' := T' - \{v_{i_1}, v_{i_2}, v_{i_3}\}$, $v_{i_j} := v_{i_{j+3}}$ for $1 \leqslant j \leqslant k - 3$, $k := k - 3$, go to Step 8.

Step 13: If $k = 4$, go to Step 14. If $k = 5$, set $S := S \cup \{v_{i_4}, v_{i_5}\}$, go to Step 14. If $k \geqslant 6$, set $T' := T' - \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\}$, $v_{i_j} := v_{i_{j+4}}$ for $1 \leqslant j \leqslant k - 4$, $k := k - 4$, go to Step 8.

Step 14: Set $T_{20} := T_{20} - V(P)$, go to Step 6.

Output: The vertex set S, which is a minimum paired-dominating set of the tree T_v.

The complexity of the above algorithm can be estimated as follows. The time used in performing Step 2 is clearly $O(|V(T_v)|)$. The time used in performing Step 4 for a given branch vertex u is $O(|C(u)|)$. Hence, the time used in the loop from Step 3 to Step 4 is at most $O(|V(T_v)|)$. The loop from Step 8 to Step 13 determines the minimum paired-dominating set of a path P under the condition that each vertex labelled $*$ must be included in the paired-dominating set. The time used is clearly at most $O(|V(P)|)$. Thus, the time used in the loop from Step 6 to Step 14 is at most $O(|V(T_v)|)$. It follows that the total time used in the performance of the above algorithm is $O(|V(T_v)|)$. We prove next the correctness of the algorithm.

By Algorithm 1, it is easily seen

PROPERTY 1. *(a) Any branch of the graph T_{20} produced by Step 1–Step 5 is a path, which has at most one end vertex labelled $*$.*

(b) For any branch P of the graph T_{20}, if $V(P) \cap J = \emptyset$, then $V(P) \cap S$ is a γ_p-set of P.

Property 1 ensures that for every branch P of T_{20} there exists a paired-dominating set of P containing all vertices of J.

THEOREM 2. *Given a tree T of order n, Algorithm 1 computes in time $O(n)$ a minimum paired-dominating set of T.*

Before we prove the theorem, we first give some lemmas which will be used in the proof.

LEMMA 1. *If v is a remote vertex of tree T, then for every paired-dominating set S, $v \in S$.*

Proof. Assume u is a leaf and $vu \in E(T)$. To dominate u either u or $v \in S$. By the definition of paired-dominating set, $u \in S$ implies $v \in S$. The result follows. □

LEMMA 2. *Let P_l be a path $v_1 v_2 \ldots v_l$, then $\gamma_p(P_l) = 2\lceil l/4 \rceil$.*

LEMMA 3. *Suppose T is a tree rooted at v and let u be a branch vertex at maximum distance from v.*
(a) If $C^0(u) \neq \emptyset$, then there exists a γ_p-set of T containing u.
(b) If $w \in C^1(u) \neq \emptyset$, then there exists a γ_p-set of T containing u and w.

Proof. (a) Let X be a γ_p-set of T. If $C^0(u) \cap L(T) \neq \emptyset$, by Lemma 1, $u \in X$. We may assume $w \in C^0(u) - L(T)$ and $u \not\in X$, then $X \cap D[w] \succ T_w$. So $|X \cap D[w]| \geqslant 2\lceil |D[w]/4 \rceil$. By Lemma 2, $\gamma_p(D[c(w)]) = 2|D[c(w)]|/4 = 2\lceil |D[w]|/4 \rceil - 2$. Let S_1 be a γ_p-set of $T_{c(w)}$, then $S_1 \cup \{w, u\} \succ T_w$. Furthermore $X_1 = (X - D[w]) \cup \{w, u\} \cup S_1$ is a γ_p-set of T and $u \in X_1$.

(b) Let X be a γ_p-set of T. If w and $u \in X$, the theorem follows. So we may assume that either $u \not\in X$ or $w \not\in X$. If $u \not\in X$, then $X \cap D[w] \succ T_w$. So $|X \cap D[w]| \geqslant 2\lceil |D[w]|/4 \rceil$. Let $w_1 \in N(c(w)) - w$, then $\gamma_p(T_{w_1}) = 2|D[w_1]|/4 = 2\lceil |D[w]|/4 \rceil - 2$. Let S_1 be a γ_p-set of T_{w_1}, then $S_1 \cup \{u, w\} \succ T_w$. Furthermore, $X_1 = (X - D[w]) \cup \{u, w\} \cup S_1$ is a γ_p-set of T and $u, w \in X_1$.

If $u \in X$, $w \notin X$, then $X \cap D[c(w)] \succ T_{c(w)}$. So $|X \cap D[c(w)]| \geqslant 2\lceil |D[c(w)]|/4 \rceil = 2|D[w_1]|/4 + 2$. Let S_1 be a γ_p-set of T_{w_1}, then $X_1 = (X - D[c(w)]) \cup \{w, c(w)\} \cup S_1$ is a γ_p-set of T and $u, w \in X_1$. □

Proof of Theorem 2. Let T be a tree on n vertices. We proceed by induction on n. Let u be a branch vertex at maximum distance from v. Let y be a child of u of lowest priority. We consider $T' = T - D[y]$. Let S be a paired-dominating set of T produced by Algorithm 1. If $y \in C^3(u) \cup C^2(u) \cup C^1(u)$, then $S' = S \cap V(T')$ is a paired-dominating set of T' produced by Algorithm 1. By induction hypothesis, S' is a γ_p-set of T'. And by Property 1(b), $S \cap D[y]$ is a γ_p-set of T_y. So $\gamma_p(T) \leqslant \gamma_p(T') + \gamma_p(T_y) = |S|$. If $y \in C^0(u)$, then $u \in S$. Let S' be a paired-dominating set of T' produced by Algorithm 1. From Algorithm 1, we know that $|S \cap V(T')| = |S'|$. By induction hypothesis, S' is a γ_p-set of T'. So $S \cap V(T')$ is a γ_p-set of T'. Then $\gamma_p(T) \leqslant \gamma_p(T') + \gamma_p(T_{c(y)}) = |S|$. Furthermore, we show that $\gamma_p(T) \geqslant |S|$ by considering four cases.

CASE 1. $y \in C^3(u)$

Let y_1 be a child of u of highest priority.

If $y_1 \in C^0(u)$, by Lemma 3, there exists a γ_p-set X of T such that $u \in X$. If $y \notin X$, then $X \cap D[y] \succ T_{c(y)}$, $X \cap V(T') \succ T'$. By Lemma 2, $\gamma_p(T_{c(y)}) = \gamma_p(T_y)$. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$. If $y \in X$ and y, u are not paired, then $X \cap V(T') \succ T'$ and $X \cap D[y] \succ T_y$. So $|X| = \gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$. If $y \in X$ and y, u are paired. Let $y' = c(y)$, then $X \cap D[y'] \succ T_{c(y')}$. By Lemma 2, $\gamma_p(T_y) = \gamma_p(T_{c(y')})$. We may assume that there exists a vertex $w \in N(u) - y$ such that $w \notin X$. Otherwise, let S_1 be a γ_p-set of T_y, then $X_1 = (X - \{u, y\} \cup D(y)) \cup S_1$ is a paired-dominating set of T, and $|X_1| < |X|$, a contradiction. Then $X' = (X - D[y]) \cup \{w\} \cup S_1$ is a γ_p-set of T and $X' \cap V(T') \succ T'$, $X' \cap D[y] \succ T_y$. So $|X'| = \gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$.

If $y_1 \in C^1(u)$, by Lemma 3, there exists a γ_p-set X of T such that $y_1, u \in X$. If $y \notin X$, then $X \cap D[y] \succ T_{c(y)}$, $X \cap V(T') \succ T'$. By Lemma 2, $\gamma_p(T_y) = \gamma_p(T_{c(y)})$. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$. If $y \in X$, we claim that u, y are not paired. Otherwise, if u, y are paired in X, let $y' = c(y)$, then $X \cap D[y'] \succ T_{c(y')}$ and $X \cap D[y_1] \succ T_{y_1}$. Let $y_2 = c(y_1)$, by Lemma 2, $\gamma_p(T_{y_1}) = \gamma_p(T_{c(y_2)}) + 2$ and $\gamma_p(T_y) = \gamma_p(T_{c(y')})$. Let S_1 be a γ_p-set of $T_{c(y_2)}$ and let S_2 be a γ_p-set of T_y, then $X' = (X - D(y_1) \cup D[y]) \cup S_1 \cup S_2$ is a paired-dominating set of T. But $|X'| < |X|$, a contradiction. So u, y are not paired. Then $X \cap V(T') \succ T'$ and $X \cap D[y] \succ T_y$. So $|X'| = \gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$.

If $y_1 \in C^2(u)$, let X be a γ_p-set of T. If either $u, y \notin X$ or $u \notin X$, $y, y_1 \in X$, then $X \cap V(T') \succ T'$, $X \cap D[y] \succ T_y$. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$. If $u, y_1 \notin X$, $y \in X$, let $y' = c(y)$, $y_2 = c(y')$, then $X \cap D[y_2] \succ T_{c(y_2)}$. By Lemma 2, $\gamma_p(T_y) = \gamma_p(T_{c(y_2)})$. Let S_1 be a γ_p-set of T_y, then $X' = (X - D[y]) \cup S_1 \cup \{u, y_1\}$ is a γ_p-set of T, and $X' \cap V(T') \succ T'$, $X' \cap D[y] \succ T_y$. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$. If $u \in X$, $y \notin X$, then $X \cap D[c(y)] \succ T_{c(y)}$, $X \cap V(T') \succ T'$. By Lemma 2, $\gamma_p(T_{c(y)})–\gamma_p(T_y)$. So

$\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$. If $u, y \in X$, and u, y are not paired, then $X \cap V(T') \succ T'$, $X \cap D[y] \succ T_y$. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$. If $u, y \in X$ and u, y are paired in X, then $X \cap D[y'] \succ T_{c(y')}$. By Lemma 2, $\gamma_p(T_{c(y')}) = \gamma_p(T_y)$. We may assume that there exists a vertex $w \in N(u) - y$ such that $w \in X$. Otherwise, a contradiction will be yielded. Let S_1 be a γ_p-set of T_y, then $X' = (X - D[y]) \cup S_1 \cup \{w\}$ is a γ_p-set of T, and $X' \cap V(T') \succ T'$, $X' \cap D[y] \succ T_y$. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$.

If $y_1 \in C^3(u)$, discussed similarly, $\gamma_p(T) \geqslant |S|$.

CASE 2. $y \in C^2(u)$

Let y_1 be a child of u of highest priority, then $y_1 \in C^0(u) \cup C^1(u) \cup C^2(u)$.

If $y_1 \in C^0(u)$, by Lemma 3, there exists a γ_p-set X such that $u \in X$. If $y \notin X$, then $X \cap D[y] \succ T_{c(y)}$, $X \cap T' \succ T'$. By Lemma 2, $\gamma_p(T_{c(y)}) = \gamma_p(T_y)$. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$. If $y \in X$ and y, u are not paired, then $X \cap D[y] \succ T_y$, $X \cap T' \succ T'$. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$. If $y \in X$ and y, u are paired. Without loss of generality, we may assume that there exists a vertex $w \in N(u) - y$ such that $w \notin X$. Let $y' = c(y)$, then $X \cap D[c(y)] \succ T_{c(y')}$. By Lemma 2, $\gamma_p(T_y) = \gamma_p(T_{c(y')})$. Let S_1 be a γ_p-set of $D[y]$, then $X' = (X - D[y]) \cup S_1 \cup \{w\}$ is a γ_p-set of T. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$.

If $y_1 \in C^1(u) \cup C^2(u)$, discussed similarly, $\gamma_p(T) \geqslant |S|$.

CASE 3. $y \in C^1(u)$

Let y_1 be a child of u of highest priority, then $y_1 \in C^1(u)$. By Lemma 3, there exists a γ_p-set X of T such that $u, y_1 \in X$. Since y_1, y have equal priority, we may assume either $y \notin X$ or $y \in X$ and y, u are not paired with a perfect matching M of $\langle X \rangle$. Then $X \cap V(T') \succ T'$, $X \cap D[y] \succ T_{c(y)}$. By Lemma 2, $\gamma_p(T_{c(y)}) = \gamma_p(T_y)$. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_y) = |S|$.

CASE 4. $y \in C^0(u)$

By Lemma 3, there exists a γ_p-set of T such that $u \in X$. Without loss of generality, we may assume $y \notin X$, then $X \cap D[c(y)] \succ T_{c(y)}$, $X \cap T' \succ T'$. So $\gamma_p(T) \geqslant \gamma_p(T') + \gamma_p(T_{c(y)}) = |S|$.

Then $\gamma_p(T) = |S|$, S is a γ_p-set of T. □

3. A characterization of (γ, γ_p)-trees

In this section, we characterize trees with equal domination and paired-domination numbers. To state the characterization, we introduce four types of operations that we use to construct trees with equal domination and paired-domination numbers.

Type-1 operation: Attach a path P_1 to a vertex of a tree T, which is in a γ_p-set of T.

Type-2 operation: Attach a P_5 to a vertex v of a tree T, where v is in a γ_p-set of T and for every γ-set X of T, there is no vertex u such that $PN(u, X) = v$ in T.

Type-3 operation: Attach a remote vertex of P_4 to a vertex v of a tree T, where v is a vertex such that for every γ-set X of T, there is no vertex u such that $PN(u, X) = v$ in T.

Type-4 operation: Attach a vertex u_0 of tree T_1 to a vertex of a tree T, where T_1 is a tree with $V(T_1) = \{u_0, u_1, u_2, u_3, u_4\}$ and $E(T_1) = \{u_0u_1, u_1u_2, u_1u_3, u_2u_4\}$.

Let J_p be the family of trees that have equal domination and paired-domination numbers. Then

$$J_p = \{T\ ;\ \gamma(T) = \gamma_p(T)\}$$

LEMMA 4. *If T is a tree with $\gamma_p(T) = \gamma(T)$, S is a γ_p-set of T, then for each $v \in S$, $PN(v, S) \neq \emptyset$.*

Proof. Suppose to the contrary that there exists a vertex $v \in S$ such that $PN(v, S) = \emptyset$, then $S - \{v\}$ is a dominating set of T, a contradiction. This completes the proof of the lemma. □

LEMMA 5. *If T is a tree with $\gamma_p(T) = \gamma(T)$, then T has a unique γ_p-set.*

Proof. We proceed by induction on n, the order of the tree T. If $n \leqslant 4$, then $T \in \{P_4\}$ and T has a unique γ_p-set. Let $n \geqslant 5$ and assume that for all trees $T' \in J_p$ of order $n', n' < n$, T' has a unique γ_p-set. Let $T \in J_p$ be a tree of order n and let $v_0, v_1, \ldots, v_t$ be a longest path in T. If $d(v_1) \geqslant 3$, then there exists a leaf u such that $v_1u \in E(T)$. Let $T' = T - u$, then by Lemma 1 we have

$$\gamma_p(T') = \gamma_p(T) = \gamma(T) = \gamma(T')\,.$$

It follows that T has a unique γ_p-set of T. Hence, we may assume that $d(v_1) = 2$.

Claim 1. For any $u \in N(v_2) - \{v_1\}$, u is not a remote vertex.

Suppose to the contrary that v_2 is adjacent to a remote vertex $u_1 \neq v_1$, then for every γ_p-set S of T, $S \cap L(T) \neq \emptyset$. And for each $v \in S \cap L(T)$, $PN(S, v) = \emptyset$. By Lemma 4, this is a contradiction.

By Claim 1, we may suppose that either v_2 is adjacent to a leaf or $d(v_2) = 2$.

CASE 1. v_2 IS ADJACENT TO A LEAF

Let S be a γ_p-set of T, then $\{v_1, v_2\} \subseteq S$. Let T_{v_3} denotes the subtree of $T - \{v_2v_3\}$ containing v_3.

If $v_3 \notin PN(v_2, S)$, then $S - \{v_1, v_2\}$ is a paired-dominating set of T_{v_3}. So $\gamma_p(T_{v_3}) \leqslant \gamma_p(T) - 2$. However, any γ_p-set of T_{v_3} can be extended to a paired-dominating set of T by adding the vertex v_1 and v_2. So $\gamma_p(T) \leqslant \gamma_p(T_{v_3}) + 2$. Consequently, $\gamma_p(T) = \gamma_p(T_{v_3}) + 2$. Since $\gamma(T) \leqslant \gamma(T_{v_3}) + 2 \leqslant \gamma_p(T_{v_3}) + 2 = \gamma_p(T)$ and $T \in J_p$, $\gamma_p(T_{v_3}) = \gamma(T_{v_3})$. Applying the inductive hypothesis to T_{v_3}, T_{v_3} has a unique paired-dominating set S_1. It follows that $S = S_1 \cup \{v_1, v_2\}$ is a unique γ_p-set of T.

If $v_3 \in PN(v_2, S)$, by Lemma 1, v_3 is not a remote vertex, and neither v_3 is adjacent to a remote nor v_3 is adjacent to a vertex which is adjacent to a remote vertex. Then $d(v_3) = 1$ or $d(v_3) = 2$. If $d(v_3) = 1$, then T has a unique γ_p-set $S = \{v_1, v_2\}$. If $d(v_3) = 2$, let $T' = T_{v_3} - \{v_3\}$, then for any γ_p-set S of T, $S - \{v_1, v_2\}$ is a paired-dominating set of T'. So $\gamma_p(T') \leqslant \gamma_p(T) - 2$. However, any γ_p-set of T' can be extended to a paired-dominating set of T by adding the vertices v_1 and v_2. So $\gamma_p(T) \leqslant \gamma_p(T') + 2$. Consequently, $\gamma_p(T) = \gamma_p(T') + 2$. Since $\gamma(T) \leqslant \gamma(T') + 2 \leqslant \gamma_p(T') + 2 = \gamma_p(T)$ and $T \in J_p$, $\gamma_p(T') = \gamma(T')$. Applying the inductive hypothesis to T', T' has a unique γ_p-dominating set S_1. It follows that $S = S_1 \cup \{v_1, v_2\}$ is a unique γ_p-set of T.

CASE 2. $d(v_2) = 2$

By Lemma 1 and Lemma 4, for any γ_p-set S of T, $v_1, v_2 \in S$, $v_3 \in PN(v_2, S)$ and $v_4 \notin S$. As discussed in Case 1, we can infer $d(v_3) = 2$. Furthermore, we will prove $d(v_4) = 2$. Otherwise, v_4 is adjacent to a vertex $u_1 (u_1 \neq v_3, v_5)$. It is easily seen that neither u_1 is a leaf nor u_1 is a remote vertex. Let T_{v_4} denotes the subtree of $T - \{v_4v_5\}$ containing v_4, and T_{v_5} denotes the subtree of $T - \{v_4v_5\}$ containing v_5. Then $\gamma(T_{v_4}) \leqslant d(v_4) < |S \cap T_{v_4}| = 2(d(v_4) - 1)$. Since $v_4 \notin S$, $S \cap T_{v_5}$ is a paired-dominating set of T_{v_5}, so $\gamma_p(T_{v_5}) \leqslant \gamma_p(T) - 2(d(v_4) - 1)$ and $\gamma(T) \leqslant \gamma(T_{v_4}) + \gamma(T_{v_5}) \leqslant d(v_4) + \gamma(T_{v_5})$. Then $\gamma_p(T) > \gamma(T)$, a contradiction. So $d(v_4) = 2$, $v_5 \in S$. Let $T' = T - \{v_0, v_1, v_2, v_3\}$, then $S - \{v_1, v_2\}$ is a paired-dominating set of T'. So $\gamma_p(T') \leqslant \gamma_p(T) - 2$. However, any γ_p-set of T$'$ can be extended to a paired-dominating set of T by adding the vertices v_1 and v_2. So $\gamma_p(T) \leqslant \gamma_p(T') + 2$. Consequently, $\gamma_p(T) = \gamma_p(T') + 2$. Since $\gamma(T) \leqslant 2 + \gamma(T') \leqslant 2 + \gamma_p(T') = \gamma_p(T)$, and $\gamma(T) = \gamma_p(T)$, then $\gamma(T') = \gamma_p(T')$. Applying the inductive hypothesis to T', T' has a unique γ_p-dominating set S_1. It follows that $S = S_1 \cup \{v_1, v_2\}$ is a unique γ_p-set of T. □

LEMMA 6. *If $T' \in J_p$ and T is obtained from T' by a Type-1 operation, then $T \in J_p$.*

Proof. Suppose T is obtained from T' by attaching a vertex u to the vertex v of T'. Let S be a γ_p-set of T' with $v \in S$, it is easily seen that S is a γ_p-set of T. So $\gamma(T) \geqslant \gamma(T') = \gamma_p(T') = \gamma_p(T) \geqslant \gamma(T)$, then $\gamma(T) = \gamma_p(T)$. So $T \in J_p$. $\square$

LEMMA 7. *If $T' \in J_p$ and T is obtained from T' by a Type-2 operation, then $T \in J_p$.*

Proof. Suppose T is obtained from T' by attaching the path $u_1u_2u_3u_4u_5$ to the vertex v in T'. Let S_1 be a γ_p-set of T' with $v \in S_1$, then $S = S_1 \cup \{u_3, u_4\}$ is a paired-dominating set of T. So $\gamma_p(T) \leqslant \gamma_p(T') + 2$. For every γ-set X of T with $u_1 \in X$, if $v \notin PN(u_1, X)$, then $\gamma(T') \leqslant \gamma(T) - 2$. If $v \in PN(u_1, X)$, then $X \cap T' \succ T' - \{v\}$, $\gamma(T' - \{v\}) \leqslant \gamma(T) - 2$. So $\gamma(T') \leqslant \gamma(T' - \{v\}) + 1 \leqslant \gamma(T) - 1$. We claim that $\gamma(T') \leqslant \gamma(T) - 2$. Otherwise $\gamma(T') = \gamma(T) - 1$, then $X' = (X \cap T') \cup \{v\}$ is a γ-set of T and $PN(v, X') = v$. A contradiction to the conditions of v. Since $T' \in J_p$, $\gamma_p(T) \leqslant \gamma_p(T') + 2 = \gamma(T') + 2 \leqslant \gamma(T) \leqslant \gamma_p(T)$. Hence $\gamma(T) = \gamma_p(T)$ and $T \in J_p$. $\square$

LEMMA 8. *If $T' \in J_p$ and T is obtained from T' by a Type-3 operation, then $T \in J_p$.*

Proof. Suppose T is obtained from T' by attaching a remote vertex u_1 of P_4 to the vertex v in T', where $P_4 = u_0u_1u_2u_3$. Let S_1 be a γ_p-set of T', then $S = S_1 \cup \{u_1, u_2\}$ is a paired-dominating set of T. So $\gamma_p(T) \leqslant \gamma_p(T') + 2$. Let X be a γ-set of T with $u_1, u_2 \in X$. If $v \notin PN(u_1, X)$, then $T' \cap X \succ T'$. So $\gamma(T') \leqslant \gamma(T) - 2$. Then $\gamma(T) \geqslant \gamma(T') + 2 = \gamma_p(T') + 2 \geqslant \gamma_p(T)$. It follows $\gamma(T) = \gamma_p(T)$, so $T \in J_p$. If $v \in PN(u_1, X)$, then $T' \cap X \succ T' - \{v\}$. Discussed as in Lemma 7, we have $\gamma(T') \leqslant \gamma(T) - 2$. Then $\gamma(T) \geqslant 2 + \gamma(T') = 2 + \gamma_p(T') \geqslant \gamma_p(T)$. Hence $\gamma(T) = \gamma_p(T)$ and $T \in J_p$. $\square$

Similarly, we have the following lemma:

LEMMA 9. *If $T' \in J_p$ and T is obtained from T' by a Type-4 operation, then $T \in J_p$.*

We now define the family F_p as

$F_p = \{T \mid T$ is obtained from P_4 by a finite sequence of operations of Type-1, Type-2, Type-3 or Type-4$\}$.

LEMMA 10. $F_p \subseteq J_p$.

Proof. Suppose that $T \in F_p$, we show that $T \in J_p$. To do this, we use induction on $s(T)$, the number of operations required to construct the tree T. If $s(T) = 0$, then $T = P_4 \in J_p$. Assume that for all trees $T' \in F_p$ with $s(T') < k$, where $k \geqslant 1$ is an integer, that $T' \in J_p$. Let $T \in F_p$ be a tree with $s(T) = k$. Then T is obtained from some tree T' by one of operations. But then $T' \in F_p$ and $s(T') < k$. Applying the inductive hypothesis to T', $T' \in J_p$. Hence by Lemma 6, Lemma 7, Lemma 8 and Lemma 9, $T \in J_p$. $\square$

LEMMA 11. $J_p \subseteq F_p$.

Proof. Suppose that $T \in J_p$. We show that $T \in F_p$. To do this, we use induction on n, the order of the tree T. If $n \leqslant 4$, then $T \in \{P_4\}$ and clearly $T \in F_p$. Assume that for all trees $T' \in J_p$ of order $n' < n$, where $n \geqslant 5$ that $T' \in F_p$. Let $T \in J_p$ be a tree of order n and let $v_0 v_1 \dots v_l$ be a longest path in T. By Lemma 5, T has a unique γ_p-set S.

CASE 1. $d((v_1) \geqslant 3$

Then there exists a leaf $u \neq v_0$ such that $uv_1 \in E(T)$. Let $T' = T - u$. By Lemma 4 and Lemma 5, T has a unique γ_p-set S with $v_1, v_2 \in S$. It is easily seen that S is also a γ_p-set and a γ-set of T'. Hence $T' \in J_p$. Applying the inductive hypothesis of T', $T' \in F_p$. Hence T is obtained from T' by a Type-1 operation. Thus $T \in F_p$.

CASE 2. $d((v_1) = 2$

Case 2.1. $d(v_2) \geqslant 3$
As discussed in Lemma 5, v_2 is not adjacent to a remote vertex. So we may assume that v_2 is adjacent to a leaf.

If $v_3 \not\in PN(v_2, S)$, let $T' = T_{v_3}$ denotes the subtree of $T - \{v_2 v_3\}$ containing v_3, then $S - \{v_1, v_2\}$ is a paired-dominating set of T'. So $\gamma_p(T') \leqslant \gamma_p(T) - 2$. And any γ-set of T' can be extended to a dominating set of T by adding the vertices v_1 and v_2. So $\gamma(T) \leqslant \gamma(T') + 2$. Hence $\gamma_p(T') \leqslant \gamma_p(T) - 2 = \gamma(T) - 2 \leqslant \gamma(T')$. Then $\gamma(T') = \gamma_p(T') = \gamma_p(T) - 2$, $T' \in J_p$. Applying the inductive hypothesis to T', $T' \in F_p$. For every γ-set S' of T', we claim that there is no vertex $u \in S'$ such that $v_3 = PN(u, S')$. Otherwise, $S = \{v_1, v_2\} \cup (S' - \{u\})$ is a dominating set of T, then $\gamma(T) \leqslant \gamma(T') + 1$. But $\gamma_p(T) = \gamma(T') + 2$ implies that $\gamma(T') \geqslant \gamma(T) - 1 = \gamma_p(T) - 1 \geqslant \gamma(T') + 1$, a contradiction. So T is obtained from T' by a Type-3 operation and a finite sequence of operations of Type-1. Thus $T \in F_p$.

If $v_3 \in PN(v_2, S)$, then $d(v_3) = 1$ or $d(v_3) = 2$. If $d(v_3) = 1$, then T is obtained from P_4 by a finite sequence of operations of Type-1. Then $T \in F_p$. If $d(v_3) = 2$, let $T' = T - \{v_0, v_1\} \cup N[v_2]$. By Lemma 1 and Lemma 4, $v_1, v_2 \in S$ and $v_3 \not\in S$, then $\gamma_p(T') \leqslant \gamma_p(T) - 2 = \gamma(T) - 2 \leqslant \gamma(T')$. So $\gamma(T') = \gamma_p(T')$. Thus $T' \in J_p$. Applying the inductive hypothesis to T', $T' \in F_p$. T is obtained from T' by a Type-4 operation and a finite sequence of operations of Type-1. Thus $T \in F_p$.

Case 2.2. $d(v_2) = 2$

By Lemma 1 and Lemma 4, $v_1, v_2 \in S$, $v_3 \in PN(v_2, S)$ and $v_4 \notin S$. Discussed as Lemma 5, we have $d(v_3) = d(v_4) = 2$. Then $v_5 \in S$. Let $T' = T - \{v_0, v_1, v_2, v_3, v_4\}$, then $S_1 = S - \{v_1, v_2\}$ is a paired-dominating set of T'. So $\gamma_p(T') \leqslant \gamma_p(T) - 2$ and $\gamma(T) \leqslant \gamma(T') + 2$. Then $\gamma_p(T') \leqslant \gamma_p(T) - 2 = \gamma(T) - 2 \leqslant \gamma(T')$. So $\gamma(T') = \gamma_p(T') = \gamma_p(T) - 2$. Thus $T' \in J_p$. By Lemma 5, $S_1 = S - \{v_1, v_2\}$ is a unique γ_p-set of T'. Applying the inductive hypothesis to T', $T' \in F_p$. As discussed in Case 2.1, for every γ-set X of T' there is no vertex u such that $v_5 = PN(u, X)$. T is obtained from T' by Type-2 operation. Thus $T \in F_p$. □

By Lemma 10 and Lemma 11, we have proved the following theorem:

THEOREM 3. *For a tree T, $\gamma(T) = \gamma_p(T)$ if and only if $T \in F_p$.*

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